Toeplitz operators on generalized Bergman spaces

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Abstract. We consider the weighted Bergman spaces $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, where we set $d\mu_\lambda(z) = c_\lambda(1-|z|^2)^\lambda \, d\tau(z)$, with τ being the hyperbolic volume measure. These spaces are nonzero if and only if $\lambda > d$. For $0 < \lambda \leq d$, spaces with the same formula for the reproducing kernel can be defined using a Sobolev-type norm. We define Toeplitz operators on these generalized Bergman spaces and investigate their properties. Specifically, we describe classes of symbols for which the corresponding Toeplitz operators can be defined as bounded operators or as a Hilbert–Schmidt operators on the generalized Bergman spaces.

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1. Introduction

1.1. Generalized Bergman spaces

Let $\mathbb{B}^d$ denote the (open) unit ball in $\mathbb{C}^d$ and let τ denote the hyperbolic volume measure on $\mathbb{B}^d$, given by

$$d\tau(z) = (1 - |z|^2)^{-(d+1)} \, dz, \tag{1.1}$$

where dz denotes the $2d$-dimensional Lebesgue measure. The measure τ is natural because it is invariant under all of the automorphisms (biholomorphic mappings) of $\mathbb{B}^d$. For $\lambda > 0$, let μ_λ denote the measure

$$d\mu_\lambda(z) = c_\lambda (1 - |z|^2)^\lambda \, d\tau(z),$$

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where c_λ is a positive constant whose value will be specified shortly. Finally, let $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ denote the (weighted) **Bergman space**, consisting of those holomorphic functions on $\mathbb{B}^d$ that are square-integrable with respect to μ_λ. (Often these are defined using the Lebesgue measure as the reference measure, but all the formulas look nicer if we use the hyperbolic volume measure instead.) These spaces carry a projective unitary representation of the group $SU(d,1)$.

If $\lambda > d$, then the measure μ_λ is finite, so that all bounded holomorphic functions are square-integrable. For $\lambda > d$, we choose c_λ so that μ_λ is a probability measure. Calculation shows that

$$c_\lambda = \frac{\Gamma(\lambda)}{\pi^d \Gamma(\lambda - d)}, \quad \lambda > d. \tag{1.2}$$

(This differs from the value in Zhu's book [Z2] by a factor of $\pi^d/d!$, because Zhu uses normalized Lebesgue whereas we use un-normalized Lebesgue measure in (1.1).) On the other hand, if $\lambda \leq d$, then μ_λ is an infinite measure. In this case, it is not hard to show that there are no nonzero holomorphic functions that are square-integrable with respect to μ_λ (no matter which nonzero value for c_λ we choose).

Although the holomorphic L^2 space with respect to μ_λ is trivial (zero dimensional) when $\lambda \leq d$, there are indications that life does not end at $\lambda = d$. First, the reproducing kernel for $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is given by

$$K_\lambda(z,w) = \frac{1}{(1 - z \cdot \bar{w})^\lambda}$$

for $\lambda > d$. The reproducing kernel is defined by the property that it is antiholomorphic in w and satisfies

$$\int_{\mathbb{B}^d} K_\lambda(z,w) f(w) \; d\mu_\lambda(w) = f(z)$$

for all $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. Nothing unusual happens to K_λ as λ approaches d. In fact, $K_\lambda(z,w) := (1 - z \cdot \bar{w})^{-\lambda}$ is a "positive definite reproducing kernel" for all $\lambda > 0$. Thus, it is possible to define a reproducing kernel Hilbert space for all $\lambda > 0$ that agrees with $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ for $\lambda > d$.

Second, in representation theory, one is sometimes led to consider spaces like $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ but with $\lambda < d$. Consider, for example, the much-studied metaplectic representation of the connected double cover of $SU(1,1) \cong Sp(1,\mathbb{R})$. This representation is a direct sum of two irreducible representations, one of which can be realized in the Bergman space $\mathcal{H}L^2(\mathbb{B}^1, \mu_{3/2})$ and the other of which can be realized in (a suitably defined version of) the Bergman space $\mathcal{H}L^2(\mathbb{B}^1, \mu_{1/2})$. To be precise, we can say that the second summand of the metaplectic representation is realized in a Hilbert space of holomorphic functions having K_λ, $\lambda = 1/2$, as its reproducing kernel. See [14, Sect. 4.6].

Last, one often wants to consider the infinite-dimensional ($d \to \infty$) limit of the spaces $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. (See, for example, [25] and [23].) To do this, one wishes to embed each space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ isometrically into a space of functions on $\mathbb{B}^{d+1}$,

as functions that are independent of z_{n+1}. It turns out that if one uses (as we do) hyperbolic volume measure as the reference measure, then the desired isometric embedding is achieved by embedding $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ into $\mathcal{H}L^2(\mathbb{B}^{d+1}, \mu_\lambda)$. That is, if we use the *same value of* λ on $\mathbb{B}^{d+1}$ as on $\mathbb{B}^d$, then the norm of a function $f(z_1, \ldots, z_d)$ is the same whether we view it as a function on $\mathbb{B}^d$ or as a function on $\mathbb{B}^{d+1}$ that is independent of z_{d+1}. (See, for example, Theorem 4, where the inner product of z^m with z^n is independent of d.) If, however, we keep λ constant as d tends to infinity, then we will eventually violate the condition $\lambda > d$.

Although it is possible to describe the Bergman spaces for $\lambda \leq d$ as reproducing kernel Hilbert spaces, this is not the most convenient description for calculation. Instead, drawing on several inter-related results in the literature, we describe these spaces as "holomorphic Sobolev spaces," also called Besov spaces. The inner product on these spaces, which we denote as $H(\mathbb{B}^d, \lambda)$, is an L^2 inner product involving both the functions and *derivatives* of the functions. For $\lambda > d$, $H(\mathbb{B}^d, \lambda)$ is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ (the same space of functions with the same inner product), but $H(\mathbb{B}^d, \lambda)$ is defined for all $\lambda > 0$.

It is worth mentioning that in the borderline case $\lambda = d$, the space $H(\mathbb{B}^d, \lambda)$ can be identified with the Hardy space of holomorphic functions that are square-integrable over the boundary. To see this, note that the normalization constant c_λ tends to zero as λ approaches d from above. Thus, the measure of any compact subset of $\mathbb{B}^d$ tends to zero as $\lambda \to d^+$, meaning that most of the mass of μ_λ is concentrated near the boundary. As $\lambda \to d^+$, μ_λ converges, in the weak-$*$ topology on $\overline{\mathbb{B}^d}$, to the unique rotationally invariant probability measure on the boundary. Alternatively, we may observe that the formula for the inner product of monomials in $H(\mathbb{B}^d, d)$ (Theorem 4 with $\lambda = d$) is the same as in the Hardy space.

1.2. Toeplitz operators

One important aspect of Bergman spaces is the theory of Toeplitz operators on them. If ϕ is a bounded measurable function, the we can define the **Toeplitz operator** T_ϕ on $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ by $T_\phi f = P_\lambda(\phi f)$, where P_λ is the orthogonal projection from $L^2(\mathbb{B}^d, \mu_\lambda)$ onto the holomorphic subspace. That is, T_ϕ consists of multiplying a holomorphic function by ϕ, followed by projection back into the holomorphic subspace. Of course, T_ϕ depends on λ, but we suppress this dependence in the notation. The function ϕ is called the (Toeplitz) **symbol** of the operator T_ϕ. The map sending ϕ to T_ϕ is known as the Berezin–Toeplitz quantization map and it (and various generalizations) have been much studied. See, for example, the early work of Berezin [5, 6], which was put into a general framework in [26, 27], along with [22, 8, 7, 10], to mention just a few works. The Berezin–Toeplitz quantization may be thought of as a generalization of the anti-Wick-ordered quantization on $\mathbb{C}^d$ (see [15]).

When $\lambda < d$, the inner product on $H(\mathbb{B}^d, \lambda)$ is not an L^2 inner product, and so the "multiply and project" definition of T_ϕ no longer makes sense. Our strategy is to find alternative formulas for computing T_ϕ in the case $\lambda > d$, with the hope that these formulas will continue to make sense (for certain classes of symbols ϕ)

for $\lambda \leq d$. Specifically, we will identify classes of symbols ϕ for which T_ϕ can be defined as:

- A bounded operator on $H(\mathbb{B}^d, \lambda)$ (Section 4)
- A Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$ (Section 5).

We also consider in Section 3 Toeplitz operators whose symbols are polynomials in z and $\bar{z}$ and observe some unusual properties of such operators in the case $\lambda < d$.

1.3. Acknowledgments

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2. $H(\mathbb{B}^d, \lambda)$ as a holomorphic Sobolev space

In this section, we construct a Hilbert space of holomorphic functions on $\mathbb{B}^d$ with reproducing kernel $(1 - z \cdot \bar{w})^{-\lambda}$, for an arbitrary $\lambda > 0$. We denote this space as $H(\mathbb{B}^d, \lambda)$. The inner product on this space is an L^2 inner product with respect to the measure $\mu_{\lambda+2n}$, where n is chosen so that $\lambda + 2n > d$. The inner product, however, involves not only the holomorphic functions but also their derivatives. That is, $H(\mathbb{B}^d, \lambda)$ is a sort of holomorphic Sobolev space (or Besov space) with respect to the measure $\mu_{\lambda+2n}$. When $\lambda > d$, our space is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$—not just the same space of functions, but also the same inner product. When $\lambda \leq d$, the Hilbert space $H(\mathbb{B}^d, \lambda)$, with the associated projective unitary action of $SU(d, 1)$, is sometimes referred to as the analytic continuation (with respect to λ) of the holomorphic discrete series.

Results in the same spirit as—and in some cases almost identical to—the results of this section have appeared in several earlier works, some of which treat arbitrary bounded symmetric domains and not just the ball in $\mathbb{C}^d$. For example, in the case of the unit ball in $\mathbb{C}^d$, Theorem 3.13 of [30] would presumably reduce to almost the same expression as in our Theorem 4, except that Yan has all the derivatives on one side, in which case the inner product has to be interpreted as a limit of integrals over a ball of radius $1 - \varepsilon$. (Compare the formula for $\mathcal{D}_\lambda^k$ on p. 13 of [30] to the formula for A and B in Theorem 4.) See also [2, 4, 21, 31, 32]. Note, however, a number of these references give a construction that yields, for $\lambda > d$, the same space of functions as $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ with a different but equivalent norm. Such an approach is not sufficient for our needs; we require the *same inner product* as well as the same space of functions.

Although our results in this section are not really new, we include proofs to make the paper self-contained and to get the precise form of the results that we want. The integration-by-parts argument we use also serves to prepare for our definition of Toeplitz operators on $H(\mathbb{B}^d, \lambda)$ in Section 4. We ourselves were introduced to this sort of reasoning by the treatment in Folland's book [14] of the

disk model for the metaplectic representation. The paper [16] obtains results in the same spirit as those of this section, but in the context of a complex semisimple Lie group.

We begin by showing that for $\lambda > d$, the space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ can be expressed as a subspace of $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$, with a Sobolev-type norm, for any positive integer n. Let N denote the "number operator," defined by

$$N = \sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}.$$

This operator satisfies $Nz^m = |m|z^m$ for all multi-indices m. If f is holomorphic, then Nf coincides with the "radial derivative" $df(rz)/dr|_{r=1}$. We use also the operator $\bar{N} = \sum_{j=1}^{d} \bar{z}_j \partial/\partial \bar{z}_j$.

A simple computation shows that

$$(1-|z|^2)^\alpha = \left(I - \frac{N}{\alpha+1}\right)(1-|z|^2)^{\alpha+1} = \left(I - \frac{\bar{N}}{\alpha+1}\right)(1-|z|^2)^{\alpha+1}. \tag{2.1}$$

We will use (2.1) and the following integration by parts result, which will also be used in Section 4.

Lemma 1. *If $\lambda > d$ and ψ is a continuously differentiable function for which ψ and $N\psi$ are bounded, then*

$$\begin{aligned} c_\lambda \int_{\mathbb{B}^d} \psi(z)(1-|z|^2)^{\lambda-d-1} dz &= c_{\lambda+1} \int_{\mathbb{B}^d} \left[\left(I + \frac{N}{\lambda}\right)\psi\right](z)(1-|z|^2)^{\lambda-d}\, dz \\ &= c_{\lambda+1} \int_{\mathbb{B}^d} \left[\left(I + \frac{\bar{N}}{\lambda}\right)\psi\right](z)(1-|z|^2)^{\lambda-d}\, dz. \end{aligned}$$

Here dz is the $2d$-dimensional Lebesgue measure on $\mathbb{B}^d$.

Proof. We start by applying (2.1) and then think of the integral over $\mathbb{B}^d$ as the limit as r approaches 1 of the integral over a ball of radius $r < 1$. On the ball of radius r, we write out $\partial/\partial z_j$ in terms of $\partial/\partial x_j$ and $\partial/\partial y_j$. For, say, the $\partial/\partial x_j$ term we express the integral as a one-dimensional integral with respect to x_j (with limits of integration depending on the other variables) followed by an integral with respect to the other variables. We then use ordinary integration by parts in the x_j integral, and similarly for the $\partial/\partial y_j$ term.

The integration by parts will yield a boundary term involving $z_j\psi(z)(1 - |z|^2)^{\lambda-d}$; this boundary term will vanish as r tends to 1, because we assume $\lambda > d$. In the nonboundary term, the operator N applied to $(1-|z|^2)^{\lambda-d}$ will turn into the operator $-\sum_{j=1}^{d} \partial/\partial z_j \circ z_j = -(dI + N)$ applied to ψ. Computing from (1.2) that $c_\lambda/c_{\lambda+1} = (\lambda-d)/\lambda$, we may simplify and let r tend to 1 to obtain the desired result involving N. The same reasoning gives the result involving $\bar{N}$ as well. □

We now state the key result, obtained from (2.1) and Lemma 1, relating the inner product in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ to the inner product in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+1})$ (compare [14, p. 215] in the case $d = 1$).

Proposition 2. *Suppose that $\lambda > d$ and f and g are holomorphic functions on $\mathbb{B}^d$ for which f, g, Nf, and Ng are all bounded. Then*

$$\langle f, g\rangle_{L^2(\mathbb{B}^d,\mu_\lambda)} = \left\langle f, \left(I + \frac{N}{\lambda}\right) g\right\rangle_{L^2(\mathbb{B}^d,\mu_{\lambda+1})} = \left\langle \left(I + \frac{N}{\lambda}\right) f, g\right\rangle_{L^2(\mathbb{B}^d,\mu_{\lambda+1})}. \tag{2.2}$$

Proof. Recalling the formula (1.1) for the measure τ, we apply Lemma 1 with $\psi(z) = \overline{f(z)}g(z)$ with f and g holomorphic. Observing that $N(\bar{f}g) = \bar{f}Ng$ gives the first equality and observing that $\bar{N}(\bar{f}g) = \overline{(Nf)}g$ gives the second equality. □

Now, a general function in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is not bounded. Indeed, the pointwise bounds on elements of $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, coming from the reproducing kernel, are not sufficient to give a direct proof of the vanishing of the boundary terms in the integration by parts in Proposition 2. Nevertheless, (2.2) does hold for all f and g in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, *provided* that one interprets the inner product as the limit as r approaches 1 of integration over a ball of radius r. (See [14, p. 215] or [30, Thm. 3.13].) We are going to iterate (2.2) to obtain an expression for the inner product on $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ involving equal numbers of derivatives on f and g. This leads to the following result.

Theorem 3. *Fix $\lambda > d$ and a non-negative integer n. Then a holomorphic function f on $\mathbb{B}^d$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ if and only if $N^l f$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for $0 \le l \le n$. Furthermore,*

$$\langle f, g\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_\lambda)} = \langle Af, Bg\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\lambda+2n})} \tag{2.3}$$

for all $f, g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, where

$$A = \left(I + \frac{N}{\lambda+n}\right)\left(I + \frac{N}{\lambda+n+1}\right)\cdots\left(I + \frac{N}{\lambda+2n-1}\right)$$
$$B = \left(I + \frac{N}{\lambda}\right)\left(I + \frac{N}{\lambda+1}\right)\cdots\left(I + \frac{N}{\lambda+n-1}\right).$$

Let us make a few remarks about this result before turning to the proof. Let $\sigma = \lambda + 2n$. It is not hard to see that $N^k f$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$ for $0 \le k \le n$ if and only if all the partial derivatives of f up to order n belong to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\mu)$, so we may describe this condition as "f has n derivatives in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$." This condition then implies that f belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\sigma-2n})$, which in turn means that $f(z)/(1-|z|^2)^n$ belongs to $L^2(\mathbb{B}^d, \mu_\sigma)$. Since $1/(1-|z|^2)^n$ blows up at the boundary of $\mathbb{B}^d$, saying that $f(z)/(1-|z|^2)^n$ belongs to $L^2(\mathbb{B}^d, \mu_\sigma)$ says that $f(z)$ has better behavior at the boundary than a typical element of $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$. We may summarize this discussion by saying that each derivative that $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$ has in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$ results, roughly speaking, in an improvement by a factor of $(1-|z|^2)$ in the behavior of f near the boundary.

This improvement is also reflected in the pointwise bounds on f coming from the reproducing kernel. If f has n derivatives in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$, then f belongs to

$\mathcal{H}L^2(\mathbb{B}^d, \mu_{\sigma-2n})$, which means that f satisfies the pointwise bounds

$$\begin{aligned} |f(z)| &\le \|f\|_{L^2(\mathbb{B}^d,\mu_{\sigma-2n})} \left(K_{\sigma-2n}(z,z)\right)^{1/2} \\ &= \|f\|_{L^2(\mathbb{B}^d,\mu_{\sigma-2n})} \left(\frac{1}{1-|z|^2}\right)^{\frac{\sigma}{2}-n}. \end{aligned} \tag{2.4}$$

These bounds are better by a factor of $(1-|z|^2)^n$ than the bounds on a typical element of $\mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)$. See also [16] for another setting in which the existence of derivatives in a holomorphic L^2 space can be related in a precise way to improved pointwise behavior of the functions.

The results of the two previous paragraphs were derived under the assumption that $\lambda = \sigma - 2n > d$. However, Theorem 4 will show that (2.4) still holds under the assumption $\lambda = \sigma - 2n > 0$.

Proof. If f and g are polynomials, then (2.3) follows from iteration of Proposition 2. Note that N is a non-negative operator on polynomials, because the monomials form an orthogonal basis of eigenvectors with non-negative eigenvalues. It is well known and easily verified that for any f in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, the partial sums of the Taylor series of f converge to f in norm. We can therefore choose polynomials f_j converging in norm to f. If we apply (2.3) with $f = g = (f_j - f_k)$ and expand out the expressions for A and B, then the positivity of N will force each of the terms on the right-hand side to tend to zero. In particular, $N^l f_j$ is a Cauchy sequences in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$, for all $0 \le l \le n$. It is easily seen that the limit of this sequence is $N^l f$; for holomorphic functions, L^2 convergence implies locally uniform convergence of the derivatives to the corresponding derivatives of the limit function. This shows that $N^l f$ is in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$. For any $f, g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, choose sequences f_j and g_k of polynomials converging to f, g. Since $N^l f_j$ and $N^l g_j$ converge to $N^l f$ and $N^l g$, respectively, plugging f_j and g_j into (2.3) and taking a limit gives (2.3) in general.

In the other direction, suppose that $N^l f$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for all $0 \le l \le n$. Let f_j denote the jth partial sum of the Taylor series of f. Then since $Nz^m = |m| z^m$ for all multi-indices m, the functions $N^l f_j$ form the partial sums of a Taylor series converging to $N^l f_j$, and so these must be the partial sums of *the* Taylor series of $N^l f$. Thus, for each l, we have that $N^l f_j$ converges to $N^l f$ in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$. If we then apply (2.3) with $f = g = f_j - f_k$, convergence of each $N^l f_j$ implies that all the terms on the right-hand side tend to zero. We conclude that f_j is a Cauchy sequence in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, which converges to some $\hat{f}$. But L^2 convergence of holomorphic functions implies pointwise convergence, so the limit in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ (i.e., $\hat{f}$) coincides with the limit in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ (i.e., f). This shows that f is in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. $\square$

Now, when $\lambda \le d$, Proposition 2.2 no longer holds. This is because the boundary terms, which involve $(1-|z|^2)^{\lambda-d}$, no longer vanish. This failure of equality is

actually a *good* thing, because if we take $f = g$, then

$$c_\lambda \int_{\mathbb{B}^d} |f(z)|^2 (1-|z|^2)^\lambda \; d\tau(z) = +\infty$$

for all nonzero holomorphic functions, no matter what positive value we assign to c_λ. (Recall that when $\lambda > d$, c_λ is chosen to make μ_λ a probability measure, but this prescription does not make sense for $\lambda \leq d$.) Although the left-hand side of (2.2) is infinite when $f = g$ and $\lambda \leq d$, the right-hand side is finite if $\lambda + 1 > d$ and, say, f is a polynomial.

More generally, for any $\lambda \leq d$, we can choose n big enough that $\lambda + 2n > d$. We then take the right-hand side of (2.3) as a definition.

Theorem 4. *For all $\lambda > 0$, choose a non-negative integer n so that $\lambda + 2n > d$ and define*

$$H(\mathbb{B}^d, \lambda) = \left\{ f \in \mathcal{H}(\mathbb{B}^d) \,\middle|\, N^k f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}),\ 0 \leq k \leq n \right\}.$$

Then the formula

$$\langle f, g \rangle_\lambda = \langle Af, Bg \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})}$$

where

$$A = \left(I + \frac{N}{\lambda+n}\right)\left(I + \frac{N}{\lambda+n+1}\right)\cdots\left(I + \frac{N}{\lambda+2n-1}\right)$$
$$B = \left(I + \frac{N}{\lambda}\right)\left(I + \frac{N}{\lambda+1}\right)\cdots\left(I + \frac{N}{\lambda+n-1}\right)$$

defines an inner product on $H(\mathbb{B}^d, \lambda)$ and $H(\mathbb{B}^d, \lambda)$ is complete with respect to this inner product.

The monomials z^m form an orthogonal basis for $H(\mathbb{B}^d, \lambda)$ and for all multi-indices l and m we have

$$\left\langle z^l, z^m \right\rangle_\lambda = \delta_{l,m} \frac{m!\Gamma(\lambda)}{\Gamma(\lambda + |m|)}.$$

Furthermore, $H(\mathbb{B}^d, \lambda)$ has a reproducing kernel given by

$$K_\lambda(z, w) = \frac{1}{(1 - z \cdot \bar{w})^\lambda}.$$

Using power series, it is easily seen that for any holomorphic function f, if $N^n f$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$, then $N^k f$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for $0 \leq k < n$.

Note that the reproducing kernel and the inner product of the monomials are independent of n. Thus, we obtain the same space of functions with the same inner product, no matter which n we use, so long as $\lambda + 2n > d$.

From the reproducing kernel we obtain the pointwise bounds given by $|f(z)|^2 \leq \|f\|_\lambda^2 (1 - |z|^2)^{-\lambda}$.

Proof. Using a power series argument, it is easily seen that if f and $N^k f$ belong to $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$, then $\langle f, N^k f\rangle_{L^2(\mathbb{B}^d,\mu_{\lambda+2n})} \geq 0$. From this, we obtain positivity of the inner product $\langle\cdot,\cdot\rangle_\lambda$. If f_j is a Cauchy sequence in $H(\mathbb{B}^d, \lambda)$, then positivity of the coefficients in the expressions for A and B imply that for $0 \leq k \leq n$, $N^k f_j$ is a Cauchy sequence in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$, which converges (as in the proof of Theorem 3) to $N^k f$. This shows that $N^k f$ is in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for each $0 \leq k \leq n$, and so $f \in H(\mathbb{B}^d, \lambda)$. Further, convergence of each $N^k f_j$ to $N^k f$ implies that f_j converges to f in $H(\mathbb{B}^d, \lambda)$.

To compute the inner product of two monomials in $H(\mathbb{B}^d, \lambda)$, we apply the definition. Since $N z^m = |m| z^m$, we obtain

$$\begin{aligned}
&\langle z^l, z^m\rangle_\lambda \\
&= \delta_{l,m} \left(\frac{\lambda+|m|}{\lambda}\right)\left(\frac{\lambda+1+|m|}{\lambda+1}\right)\cdots\left(\frac{\lambda+2n-1+|m|}{\lambda+2n-1}\right)\frac{m!\Gamma(\lambda+2n)}{\Gamma(\lambda+2n+|m|)} \\
&= \delta_{l,m}\frac{m!\Gamma(\lambda)}{\Gamma(\lambda+|m|)},
\end{aligned}$$

where we have used the known formula for the inner product of monomials in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ (e.g., [Z2]).

Completeness of the monomials holds in $H(\mathbb{B}^d, \lambda)$ for essentially the same reason it holds in the ordinary Bergman spaces. For $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, expand f in a Taylor series and then consider $\langle z^m, f\rangle_\lambda$. Each term in the inner product is an integral over $\mathbb{B}^d$ with respect to $\mu_{\lambda+2n}$, and each of these integrals can be computed as the limit as r tends to 1 of integrals over a ball of radius $r < 1$. On the ball of radius r, we may interchange the integral with the sum in the Taylor series. But distinct monomials are orthogonal not just over $\mathbb{B}^d$ but also over the ball of radius r, as is easily verified. The upshot of all of this is that $\langle z^m, f\rangle_\lambda$ is a nonzero multiple of the mth Taylor coefficient of f. Thus if $\langle z^m, f\rangle_\lambda = 0$ for all m, f is identically zero.

Finally, we address the reproducing kernel. Although one can use essentially the same argument as in the case $\lambda > d$, using the orthogonal basis of monomials and a binomial expansion (see the proof of Theorem 12), it is more enlightening to relate the reproducing kernel in $H(\mathbb{B}^d, \lambda)$ to that in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$. We require some elementary properties of the operators A and B; since the monomials form an orthogonal basis of eigenvectors for these operators, these properties are easily obtained. We need that A is self-adjoint on its natural domain and that A and B have bounded inverses.

Let $\chi_z^{\lambda+2n}$ be the unique element of $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for which

$$\langle \chi_z^{\lambda+2n}, f\rangle_{L^2(\mathbb{B}^d,\mu_{\lambda+2n})} = f(z)$$

for all f in $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})$. Explicitly, $\chi_z^{\lambda+2n}(w) = (1 - \bar{z}\cdot w)^{-(\lambda+2n)}$. (This is Theorem 2.2 of [Z2] with our λ corresponding to $n + \alpha + 1$ in [Z2].) Now, a simple

calculation shows that

$$(I + N/a)(1 - \bar{z} \cdot w)^{-a} = (1 - \bar{z} \cdot w)^{-(a+1)}, \tag{2.5}$$

where N acts on the w variable with z fixed. From this, we see that $N^k \chi_z^{\lambda+2n}$ is a bounded function for each fixed $z \in \mathbb{B}^d$ and $k \in \mathbb{N}$, so that $\chi_z^{\lambda+2n}$ is in $H(\mathbb{B}^d, \lambda)$.

For any $f \in H(\mathbb{B}^d, \lambda)$ we compute that

$$\begin{aligned}\left\langle f, (AB)^{-1} \chi_z^{\lambda+2n} \right\rangle_\lambda &= \left\langle Af, B(AB)^{-1} \chi_z^{\lambda+2n} \right\rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})} \\ &= \left\langle f, \chi_z^{\lambda+2n} \right\rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})} = f(z).\end{aligned}$$

This shows that the reproducing kernel for $H(\mathbb{B}^d, \lambda)$ is given by $K_\lambda(z, w) = [(AB)^{-1} \chi_z^{\lambda+2n}](w)$. Using (2.5) repeatedly gives the desired result. $\square$

We conclude this section with a simple lemma that will be useful in Section 5.

Lemma 5. *For all $\lambda_1, \lambda_2 > 0$, if f is in $H(\mathbb{B}^d, \lambda_1)$ and g is in $H(\mathbb{B}^d, \lambda_2)$ then fg is in $H(\mathbb{B}^d, \lambda_1 + \lambda_2)$.*

Proof. If, say, $\lambda_1 > d$, then we have the following simple argument:

$$\begin{aligned}\|fg\|_{\lambda_1+\lambda_2}^2 &= c_{\lambda_1+\lambda_2} \int_{\mathbb{B}^d} |f(z)|^2 \, |g(z)|^2 \, (1 - |z|^2)^{\lambda_1+\lambda_2} \, d\tau(z) \\ &\leq c_{\lambda_1+\lambda_2} \|g\|_{\lambda_2}^2 \int_{\mathbb{B}^d} |f(z)|^2 \, (1 - |z|^2)^{-\lambda_2} (1 - |z|^2)^{\lambda_1+\lambda_2} \, d\tau(z) \\ &= \frac{c_{\lambda_1+\lambda_2}}{c_{\lambda_1}} \|f\|_{\lambda_1}^2 \, \|g\|_{\lambda_2}^2 \, .\end{aligned}$$

Unfortunately, $c_{\lambda_1+\lambda_2}/c_{\lambda_1}$ tends to infinity as λ_1 approaches d from above, so we cannot expect this simple inequality to hold for $\lambda_1 < d$.

For any $\lambda_1, \lambda_2 > 0$, choose n so that $\lambda_1 + n > d$ and $\lambda_2 + n > d$. Then fg belongs to $H(\mathbb{B}^d, \lambda_1 + \lambda_2)$ provided that $N^n(fg)$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \lambda_1 + \lambda_2 + 2n)$. But

$$N^n(fg) = \sum_{k=0}^{n} \binom{n}{k} N^k f \; N^{n-k} g. \tag{2.6}$$

Using Theorem 4, it is easy to see that if f belongs to $H(\mathbb{B}^d, \lambda_1)$ then $N^k f$ belongs to $H(\mathbb{B}^d, \lambda_1 + 2k)$. Thus,

$$\left|N^k f(z)\right|^2 \leq a_k (1 - |z|^2)^{-(\lambda_1+2k)}.$$

Now, for each term in (2.6) with $k \leq n/2$, we then obtain the following norm estimate:

$$\begin{aligned}&c_{\lambda_1+\lambda_2+2n} \int_{\mathbb{B}^d} \left|N^k f(z) N^{n-k} g(z)\right|^2 \; (1 - |z|^2)^{\lambda_1+\lambda_2+2n} \, d\tau(z) \\ &\leq c_{\lambda_1+\lambda_2+2n} a_k \int_{\mathbb{B}^d} \left|N^{n-k} g(z)\right|^2 (1 - |z|^2)^{\lambda_2+2n-2k} \, d\tau(z).\end{aligned} \tag{2.7}$$

Since $k \leq n/2$, we have $\lambda_2 + 2n - 2k \geq \lambda_2 + n > d$. We are assuming that g is in $H(\mathbb{B}^d, \lambda_2)$, so that $N^{n-k}g$ is in $H(\mathbb{B}^d, \lambda_2 + 2n - 2k)$, which coincides with $\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda_2+2n-2k})$. Thus, under our assumptions on f and g, each term in (2.6) with $k \leq n/2$ belongs to $\mathcal{H}L^2(\mathbb{B}^d, \lambda_1 + \lambda_2 + 2n)$. A similar argument with the roles of f and g reversed takes care of the terms with $k \geq n/2$. □

3. Toeplitz operators with polynomial symbols

In this section, we will consider our first examples of Toeplitz operators on generalized Bergman spaces, those whose symbols are (not necessarily holomorphic) polynomials. Such examples are sufficient to see some interesting new phenomena, that is, properties of ordinary Toeplitz operator that fail when extended to these generalized Bergman spaces. The definition of Toeplitz operators for the case of polynomial symbols is consistent with the definition we use in Section 4 for a larger class of symbols.

For $\lambda > d$, we define the Toeplitz operator T_ϕ by

$$T_\phi f = P_\lambda(\phi f)$$

for all f in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ and all bounded measurable functions ϕ. Recall that P_λ is the orthogonal projection from $L^2(\mathbb{B}^d, \tau)$ onto the holomorphic subspace. Because P_λ is a self-adjoint operator on $L^2(\mathbb{B}^d, \mu_\lambda)$, the matrix entries of T_ϕ may be calculated as

$$\langle f_1, T_\phi f_2 \rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_\lambda)} = \langle f_1, \phi f_2 \rangle_{L^2(\mathbb{B}^d,\mu_\lambda)}, \quad \lambda > d, \tag{3.1}$$

for all $f_1, f_2 \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. From this formula, it is easy to see that $T_{\bar{\phi}} = (T_\phi)^*$.

If ψ is a bounded holomorphic function and ϕ is any bounded measurable function, then it is easy to see that $T_{\phi\psi} = T_\phi M_\psi$. Thus, for any two multi-indices m and n, we have

$$T_{\bar{z}^m z^n} = (M_{z^m})^*(M_{z^n}). \tag{3.2}$$

We will take (3.2) as a definition for $0 < \lambda \leq d$. Our first task, then, is to show that M_{z^n} is a bounded operator on $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ for all $\lambda > 0$.

Proposition 6. *For all $\lambda > 0$ and all multi-indices n, the multiplication operator M_{z^n} is a bounded operator on $H(\mathbb{B}^d, \lambda)$. Thus, for any polynomial ϕ, the Toeplitz operator T_ϕ defined in (3.2) is a bounded operator on $H(\mathbb{B}^d, \lambda)$.*

Proof. The result is a is a special case of a result of Arazy and Zhang [3] and also of the results of Section 4, but it is easy to give a direct proof. It suffices to show that M_{z_j} is bounded for each j. Since M_{z_j} preserves the orthogonality of the monomials, we obtain

$$\left\| M_{z_j} \right\| = \sup_m \frac{\left\| z_j z^m \right\|_\lambda}{\left\| z^m \right\|_\lambda} = \sup_m \frac{m_j + 1}{|m| + \lambda}.$$

Note that $m_j \leq |m|$ with equality when $m_k = 0$ for $k \neq j$. Thus the supremum is finite and is easily seen to have the value of 1 if $\lambda \geq 1$ and $1/\lambda$ if $\lambda < 1$. □

We now record some standard properties of Toeplitz operators on (ordinary) Bergman spaces. These properties hold for Toeplitz operators (defined by the "multiply and project" recipe) on any holomorphic L^2 space. We will show that these properties *do not* hold for Toeplitz operators with polynomial symbols on the generalized Bergman spaces $H(\mathbb{B}^d, \lambda)$, $\lambda < d$.

Proposition 7. *For $\lambda > d$ and $\phi(z)$ bounded, the Toeplitz operator T_ϕ on the space $\mathcal{H}L^2(\mathbb{B}^d, d\mu_\lambda)$, which is defined by $T_\phi f = P_\lambda(\phi f)$, has the following properties.*

1. $\|T_\phi\| \leq \sup_z |\phi(z)|$
2. *If $\phi(z) \geq 0$ for all z, then T_ϕ is a positive operator.*

Both of these properties fail when $\lambda < d$. In fact, for $\lambda < d$, there is no constant C such that $\|T_\phi\| \leq C \sup_z |\phi(z)|$ for all polynomials ϕ.

As we remarked in the introduction, when $\lambda = d$, the space $H(\mathbb{B}^d, \lambda)$ may be identified with the Hardy space. Thus Properties 1 and 2 in the proposition still hold when $\lambda = d$, if, say, ϕ is continuous up to the boundary of $\mathbb{B}^d$ (or otherwise has a reasonable extension to the closure of $\mathbb{B}^d$).

Proof. When $\lambda > d$, the projection operator P_λ has norm 1 and the multiplication operator M_ϕ has norm equal to $\sup_z |\phi(z)|$ as an operator on $L^2(\mathbb{B}^d, \mu_\lambda)$. Thus, the restriction to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ of $P_\lambda M_\phi$ has norm at most $\sup_z |\phi(z)|$. Meanwhile, if ϕ is non-negative, then from (3.1) we see that $\langle f, T_\phi f\rangle \geq 0$ for all $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$.

Let us now assume that $0 < \lambda < d$. Computing on the orthogonal basis in Theorem 4, it is a simple exercise to show that

$$T_{\bar{z}_j z_j}(z^m) = \frac{\Gamma(\lambda + |m|)}{m!} \frac{(m + e_j)!}{\Gamma(\lambda + |m| + 1)} z^m = \frac{1 + m_j}{\lambda + |m|} z^m. \tag{3.3}$$

If we take $\phi(z) = |z|^2$, then summing (3.3) on j gives

$$T_\phi z^m = \frac{d + |m|}{\lambda + |m|} z^m.$$

Since $\lambda < d$, this shows that $\|T_\phi\| > 1$, even though $|\phi(z)| < 1$ for all $z \in \mathbb{B}^d$. Thus, Property 1 fails for $\lambda < d$. (From this calculation it easily follows that if $\phi(z) = (1 - |z|^2)/(\lambda - d)$, then T_ϕ is the bounded operator $(\lambda I + N)^{-1}$, for all $\lambda \neq d$.)

For the second property, we let $\psi(z) = 1 - \phi(z) = 1 - |z|^2$ which is positive. From the above calculation we obtain

$$\langle T_\psi z^m, z^m\rangle_{H_\lambda} = \|z^m\|^2_{H_\lambda} - \left(\frac{d + |m|}{\lambda + |m|}\right) \|z^m\|^2_{H(\mathbb{B}^d, \lambda)},$$

which is negative if $0 < \lambda < d$.

We now show that there is no constant C such that $\|T_\phi\| \leq C \sup_z |\phi(z)|$. Consider

$$\begin{aligned}\phi_k(z) &:= (|z|^2)^k = \left(\sum_{i=1}^{d} |z_i|^2\right)^k \\ &= \sum_{|i|=k} \frac{k!}{i!}(|z_1|^2)^{i_1}(|z_2|^2)^{i_2}\cdots(|z_d|^2)^{i_d} = \sum_{|i|=k}\frac{k!}{i!}\overline{z}^i z^i.\end{aligned}$$

Computing on the orthogonal basis in Theorem 4 we obtain

$$T_{\phi_k}\mathbf{1} = \sum_{|i|=k}\frac{k!}{i!}(T_{\overline{z}^i z^i}\mathbf{1}) = \sum_{|i|=k}\frac{k!}{i!}\frac{i!\Gamma(\lambda)}{\Gamma(\lambda+k)}\mathbf{1} = \mathcal{I}\frac{k!\Gamma(\lambda)}{\Gamma(\lambda+k)}\mathbf{1},$$

where $\mathbf{1}$ is the constant function. Here, $\mathcal{I}$ is the number of multi-indices i of length d such that $|i| = k$, which is equal to $\binom{k+d-1}{d-1}$. Thus

$$T_{\phi_k}\mathbf{1} = \frac{(k+d-1)!}{(d-1)!}\frac{\Gamma(\lambda)}{\Gamma(\lambda+k)}\mathbf{1} = \frac{(d+k-1)\cdots(d)}{(\lambda+k-1)\cdots(\lambda)}\mathbf{1} = \prod_{j=0}^{k-1}\frac{d+j}{\lambda+j}\mathbf{1}.$$

Consider $\prod_{j=0}^{k-1}\frac{d+j}{\lambda+j} = \prod_{j=0}^{k-1}\left(1+\frac{d-\lambda}{\lambda+j}\right)$. Since $d > \lambda$, the terms $\frac{d-\lambda}{\lambda+j}$ are positive and $\sum_{j=0}^{\infty}\frac{d-\lambda}{\lambda+j}$ diverges. This implies $\prod_{j=0}^{\infty}\frac{d+j}{\lambda+j} = \infty$. Since $\sup_z|\phi_k(z)| = 1$ for all k, there is no a constant C such that $\|T_\phi\| \leq C\sup_z|\phi(z)|$. □

Remark 8. *For $\lambda < d$, there does not exist any positive measure ν on $\mathbb{B}^d$ such that $\|f\|_\lambda = \|f\|_{L^2(\mathbb{B}^d,\nu)}$ for all f in $H(\mathbb{B}^d,\lambda)$. If such a ν did exist, then the argument in the first part of the proof of Proposition 7 would show that Properties 1 and 2 in the proposition hold.*

4. Bounded Toeplitz operators

In this section, we will consider a class of symbols ϕ for which we will be able to define a Toeplitz operator T_ϕ as a bounded operator on $H(\mathbb{B}^d,\lambda)$ for all $\lambda > 0$. Our definition of T_ϕ will agree (for the relevant class of symbols) with the usual "multiply and project" definition for $\lambda > d$. In light of the examples in the previous section, we cannot expect boundedness of ϕ to be sufficient to define T_ϕ as a bounded operator. Instead, we will consider functions ϕ for which ϕ and a certain number of derivatives of ϕ are bounded.

Our strategy is to use integration by parts to give an alternative expression for the matrix entries of a Toeplitz operator with sufficiently regular symbol, in the case $\lambda > d$. We then take this expression as our definition of Toeplitz operator in the case $0 < \lambda \leq d$.

Theorem 9. *Assume $\lambda > d$ and fix a positive integer n. Let ϕ be a function that is $2n$ times continuously differentiable and for which $\bar{N}^k N^l \phi$ is bounded for all $0 \leq k, l \leq n$. Then*

$$\langle f, T_\phi g \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)} = c_{\lambda+2n} \int_{\mathbb{B}^d} C \left[\left(\overline{f(z)} \phi(z) g(z) \right) \right] \left(1 - |z|^2\right)^{\lambda+2n} d\tau(z)$$

for all $f, g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, where C is the operator given by

$$C = \left(I + \frac{\bar{N}}{\lambda + 2n - 1} \right) \cdots \left(I + \frac{\bar{N}}{\lambda + n} \right) \left(I + \frac{N}{\lambda + n - 1} \right) \cdots \left(I + \frac{N}{\lambda} \right). \quad (4.1)$$

Thus, there exist constants A_{jklm} (depending on n and λ) such that

$$\langle f, T_\phi g \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)} = \sum_{j,k,l,m=1}^{n} A_{jklm} \left\langle N^j f, \left(\bar{N}^k N^l \phi \right) N^m g \right\rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})}. \quad (4.2)$$

Proof. Assume at first that f and g are polynomials, so that f and g and all of their derivatives are bounded. We use (3.1) and apply the first equality in Lemma 1 with $\psi = \bar{f}\phi g$. We then apply the first equality in the lemma again with $\psi = (I + N/\lambda)[\bar{f}\phi g]$. We continue on in this fashion until we have applied the first equality in Lemma 1 n times and the second equality n times. This establishes the desired equality in the case that f and g are polynomials. For general f and g in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, we approximate by sequences f_a and g_a of polynomials. From Theorem 3 we can see that convergence of f_a and g_a in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ implies convergence of $N^j f_a$ and $N^k g_a$ to $N^j f$ and $N^k g$, so that applying (4.2) to f_a and g_a and taking a limit establishes the desired result for f and g. □

Definition 10. *Assume $0 < \lambda \leq d$ and fix a positive integer n such that $\lambda + 2n > d$. Let ϕ be a function that is $2n$ times continuously differentiable and for which $\bar{N}^k N^l \phi$ is bounded for all $0 \leq k, l \leq n$. Then we define the Toeplitz operator T_ϕ to be the unique bounded operator on $H(\mathbb{B}^d, \lambda)$ whose matrix entries are given by*

$$\langle f, T_\phi g \rangle_{H(\mathbb{B}^d, \lambda)} = c_{\lambda+2n} \int_{\mathbb{B}^d} C \left[\left(\overline{f(z)} \phi(z) g(z) \right) \right] \left(1 - |z|^2\right)^{\lambda+2n} dz, \quad (4.3)$$

where C is given by (4.1).

Note that from Theorem 4, $N^j f$ and $N^m g$ belong to $L^2(\mathbb{B}^d, \mu_{\lambda+2n})$ for all $0 \leq j, m \leq n$, for all f and g in $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. Furthermore, $\left\| N^j f \right\|_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})}$ and $\| N^m g \|_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})}$ are bounded by constants times $\|f\|_\lambda$ and $\|g\|_\lambda$, respectively. Thus, the right-hand side of (4.3) is a continuous sesquilinear form on $H(\mathbb{B}^d, \lambda)$, which means that there is a unique bounded operator T_ϕ whose matrix entries are given by (4.3).

If $\lambda = d$, then (as discussed in the introduction) the Hilbert space $H(\mathbb{B}^d, \lambda)$ is the Hardy space of holomorphic functions that are square-integrable over the boundary. In that case, the Toeplitz operator T_ϕ will be the zero operator whenever ϕ is identically zero on the boundary of $\mathbb{B}^d$. If $\lambda = d - 1$, $d - 2$, $\ldots$, then the inner product on $H(\mathbb{B}^d, \lambda)$ can be related to the inner product on the Hardy space. It is

not hard to see that in these cases, T_ϕ will be the zero operator if ϕ and enough of its derivatives vanish on the boundary of $\mathbb{B}^d$.

Let us consider the case in which $\phi(z) = \overline{\psi_1(z)}\psi_2(z)$, where ψ_1 and ψ_2 are holomorphic functions such that the function and the first n derivatives are bounded. Then when applying C to $\overline{f(z)}\phi(z)g(z)$, all the N-factors go onto the expression $\psi_2(z)g(z)$ and all the $\bar{N}$-factors go onto $\overline{f(z)\psi_1(z)}$. Recalling from Theorem 4 the formula for the inner product on $H(\mathbb{B}^d, \lambda)$, we see that

$$\langle f, T_\phi g\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_\lambda)} = \langle \psi_1 f, \psi_2 g\rangle_{H(\mathbb{B}^d,\lambda)},$$

as expected. This means that in this case, $T_{\bar{\psi}_1\psi_2} = (M_{\psi_1})^*(M_{\psi_2})$, as in the case $\lambda > d$. In particular, Definition 10 agrees with the definition we used in Section 3 in the case that ϕ is a polynomial in z and $\bar{z}$.

5. Hilbert–Schmidt Toeplitz operators

5.1. Statement of results

In this section, we will give sufficient conditions under which a Toeplitz operator T_ϕ can be defined as a Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$. Specifically, if ϕ belongs to $L^2(\mathbb{B}^d, \tau)$ then T_ϕ can be defined as a Hilbert–Schmidt operator, *provided* that $\lambda > d/2$. Meanwhile, if ϕ belongs to $L^1(\mathbb{B}^d, \tau)$, then T_ϕ can be defined as a Hilbert–Schmidt operator for all $\lambda > 0$. In both cases, we define T_ϕ in such a way that for all bounded functions f and g in $H(\mathbb{B}^d, \lambda)$, we have

$$\langle f, T_\phi g\rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} \overline{f(z)}\phi(z)g(z)(1 - |z|^2)^\lambda \, d\tau(z), \tag{5.1}$$

where c_λ is defined by $c_\lambda = \Gamma(\lambda)/(\pi^d\Gamma(\lambda - d))$. This expression is identical to (3.1) in the case $\lambda > d$. The value of c_λ should be interpreted as 0 when $\lambda - d = 0, -1, -2, \ldots$. This means that for ϕ in $L^2(\mathbb{B}^d, \tau)$ or $L^1(\mathbb{B}^d, \tau)$ (but not for other classes of symbols!), T_ϕ is the zero operator when $\lambda = d, d - 1, \ldots$. This strange phenomenon is discussed in the next subsection. Note that we are *not* claiming $T_\phi = 0$ for arbitrary symbols when $\lambda = d, d-1, \ldots$, but only for symbols that are integrable or square-integrable with respect to the hyperbolic volume measure τ. Such functions must have reasonable rapid decay (in an average sense) near the boundary of $\mathbb{B}^d$.

In the case $\phi \in L^2(\mathbb{B}^d, \tau)$, the restriction $\lambda > d/2$ is easy to explain: the function $(1 - |z|^2)^\lambda$ belongs to $L^2(\mathbb{B}^d, \tau)$ if and only if $\lambda > d/2$. Thus, if f and g are bounded and ϕ is in $L^2(\mathbb{B}^d, \tau)$, then (5.1) is absolutely convergent for $\lambda > d/2$.

In this subsection, we state our results; in the next subsection, we discuss some unusual properties of T_ϕ for $\lambda < d$; and in the last subsection of this section we give the proofs.

We begin by considering symbols ϕ in $L^2(\mathbb{B}^d, \tau)$.

Theorem 11. *Fix $\lambda > d/2$ and let $c_\lambda = \Gamma(\lambda)/(\pi^d\Gamma(\lambda - d))$. (We interpret c_λ to be zero if λ is an integer and $\lambda \leq d$.) Then the operator A_λ given by*

$$A_\lambda\phi(z) = c_\lambda^2 \int_{\mathbb{B}^d} \left[\frac{(1-|z|^2)(1-|w|^2)}{(1-w\cdot\bar{z})(1-\bar{w}\cdot z)}\right]^\lambda \phi(w)\, d\tau(w)$$

is a bounded operator from $L^2(\mathbb{B}^d, \tau)$ to itself.

Theorem 12. *Fix $\lambda > d/2$. Then for each $\phi \in L^2(\mathbb{B}^d, \tau)$, there is a unique Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$, denoted T_ϕ, with the property that*

$$\langle f, T_\phi g\rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} \overline{f(z)}\phi(z)g(z)(1-|z|^2)^\lambda\, d\tau(z) \tag{5.2}$$

for all bounded holomorphic functions f and g in $H(\mathbb{B}^d, \lambda)$. The Hilbert–Schmidt norm of T_ϕ is given by

$$\|T_\phi\|_{HS}^2 = \langle \phi, A_\lambda\phi\rangle_{L^2(\mathbb{B}^d,\tau)}.$$

If $\lambda > d$ and $\phi \in L^2(\mathbb{B}^d, \tau) \cap L^\infty(\mathbb{B}^d, \tau)$, then the definition of T_ϕ in Theorem 12 agrees with the "multiply and project" definition; compare (3.1).

Applying Lemma 5 with $\lambda_1 = \lambda_2 = \lambda$ and $\lambda > d/2$, we see that for all f and g in $H(\mathbb{B}^d, \lambda)$, the function $z \to \overline{f(z)}g(z)(1-|z|^2)^\lambda$ is in $L^2(\mathbb{B}^d, \tau)$. This means that the integral on the right-hand side of (5.2) is absolutely convergent for all $f, g \in H(\mathbb{B}^d, \lambda)$. It is then not hard to show that (5.2) holds for all $f, g \in H(\mathbb{B}^d, \lambda)$.

The operator A_λ coincides, up to a constant, with the Berezin transform. Let $\chi_z^\lambda(w) := K_\lambda(z, w)$ be the coherent state at the point z, which satisfies $f(z) = \langle \chi_z^\lambda, f\rangle_\lambda$ for all $f \in H(\mathbb{B}^d, \lambda)$. Then one standard definition of the Berezin transform B_λ is

$$B_\lambda\phi = \frac{\langle \chi_z^\lambda, T_\phi \chi_z^\lambda\rangle_\lambda}{\langle \chi_z^\lambda, \chi_z^\lambda\rangle_\lambda}.$$

The function $B_\lambda\phi$ may be thought of as the Wick-ordered symbol of T_ϕ, where T_ϕ is thought of as the *anti*-Wick-ordered quantization of ϕ. Using the formula (Theorem 4) for the reproducing kernel along with (5.2), we see that $A_\lambda = c_\lambda B_\lambda$. (Note that $\chi_z^\lambda(w)$ is a bounded function of w for each fixed $z \in \mathbb{B}^d$ and that $\langle \chi_z^\lambda, \chi_z^\lambda\rangle_\lambda = K_\lambda(z, z)$.)

Note that τ is an infinite measure, which means that if ϕ is in $L^2(\mathbb{B}^d, \tau)$ or $L^1(\mathbb{B}^d, \tau)$, then ϕ must tend to zero at the boundary of $\mathbb{B}^d$, at least in an average sense. This decay of ϕ is what allows (5.2) to be a convergent integral. If, for example, we want to take $\phi(z) \equiv 1$, then we cannot use (5.2) to define T_ϕ, but must instead use the definition from Section 3 or Section 4.

Note also that the space of Hilbert–Schmidt operators on $H(\mathbb{B}^d, \lambda)$ may be viewed as the quantum counterpart of $L^2(\mathbb{B}^d, \tau)$. It is thus natural to investigate the question of when the Berezin–Toeplitz quantization maps $L^2(\mathbb{B}^d, \tau)$ into the Hilbert–Schmidt operators.

We now show that if one considers a symbol ϕ in $L^1(\mathbb{B}^d, \tau)$, then one obtains a Hilbert–Schmidt Toeplitz operator T_ϕ for all $\lambda > 0$.

Theorem 13. *Fix $\lambda > 0$ and let c_λ be as in Theorem 12. Then for each $\phi \in L^1(\mathbb{B}^d, \tau)$, there exists a unique Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$, denoted T_ϕ, with the property that*

$$\langle f, T_\phi g\rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} \overline{f(z)}\phi(z)g(z)(1-|z|^2)^\lambda \, d\tau(z) \tag{5.3}$$

for all bounded holomorphic functions f and g in $H(\mathbb{B}^d, \lambda)$. The Hilbert–Schmidt norm of T_ϕ satisfies

$$\|T_\phi\|_{HS} \leq c_\lambda \|\phi\|_{L^1(\mathbb{B}^d,\tau)} .$$

Using the pointwise bounds on elements of $H(\mathbb{B}^d, \lambda)$ coming from the reproducing kernel, we see immediately that for all $f, g \in H(\mathbb{B}^d, \lambda)$, the function $z \to \overline{f(z)}g(z)(1-|z|^2)^\lambda$ is bounded. It is then not hard to show that (5.3) holds for all $f, g \in H(\mathbb{B}^d, \lambda)$.

We have already remarked that the definition of T_ϕ given in this section agrees with the "multiply and project" definition when $\lambda > d$ (and ϕ is bounded). It is also easy to see that the definition of T_ϕ given in this section agrees with the one in Section 4, when ϕ falls under the hypotheses of both Definition 10 and either Theorem 12 or Theorem 13. For some positive integer n, consider the set of λ's for which $\lambda + 2n > d$ and $\lambda > d/2$, i.e., $\lambda > \max(d-2n, d/2)$. Now suppose that ϕ belongs to $L^2(\mathbb{B}^d, \tau)$ and that $N^k \bar{N}^l \phi$ is bounded for all $0 \leq k, l \leq n$. It is easy to see that the matrix entries $\langle f, T_\phi g\rangle_\lambda$ depend real-analytically on λ for fixed polynomials f and g, whether T_ϕ is defined by Definition 10 or by Theorem 12. For $\lambda > d$, the two matrix entries agree because both definitions of T_ϕ agree with the "multiply and project" definition. The matrix entries therefore must agree for all $\lambda > \max(d-2n, d/2)$. Since polynomials are dense in $H(\mathbb{B}^d, \lambda)$ and both definitions of T_ϕ give bounded operators, the two definitions of T_ϕ agree. The same reasoning shows agreement of Definition 10 and Theorem 13.

5.2. Discussion

Before proceeding on with the proof, let us make a few remarks about the way we are defining Toeplitz operators in this section. For $\lambda > d$, c_λ is the normalization constant that makes the measure μ_λ a probability measure, which can be computed to have the value $\Gamma(\lambda)/(\pi^d \Gamma(\lambda - d))$. For $\lambda \leq d$, although the measure $(1-|z|^2)^\lambda \, d\tau(z)$ is an infinite measure, we simply use the same formula for c_λ in terms of the gamma function. We understand this to mean that $c_\lambda = 0$ whenever λ is an integer in the range $(0, d]$. It also means that c_λ is negative when $d-1 < \lambda < d$ and when $d-3 < \lambda < d-2$, etc.

In the cases where $c_\lambda = 0$, we have that $T_\phi = 0$ for all ϕ in $L^1(\mathbb{B}^d, \tau)$ or $L^2(\mathbb{B}^d, \tau)$. This first occurs when $\lambda = d$. Recall that for $\lambda = d$, the space $H(\mathbb{B}^d, \lambda)$ can be identified with the Hardy space of holomorphic functions square-integrable over the boundary. Meanwhile, having ϕ being integrable or square-integrable with respect to τ means that ϕ tends to zero (in an average sense) at the boundary, in which case it is reasonable that T_ϕ should be zero as an operator on the Hardy

space. For other integer values of $\lambda \leq d$, the inner product on $H(\mathbb{B}^d, \lambda)$ can be expressed using the methods of Section 2 in terms of integration over the boundary, but involving the functions and their derivatives. In that case, we expect T_ϕ to be zero if ϕ has sufficiently rapid decay at the boundary, and it is reasonable to think that having ϕ in L^1 or L^2 with respect to τ constitutes sufficiently rapid decay. Note, however, that the conclusion that $T_\phi = 0$ when $c_\lambda = 0$ applies *only* when ϕ is in L^1 or L^2; for other classes of symbols, such as polynomials, T_ϕ is not necessarily zero. For example, T_{z^m} is equal to M_{z^m}, which is certainly a nonzero operator on $H(\mathbb{B}^d, \lambda)$, for all $\lambda > 0$.

Meanwhile, if $c_\lambda < 0$, then we have the curious situation that if ϕ is positive and in L^1 or L^2 with respect to τ, then the operator T_ϕ is actually a *negative* operator. This is merely a dramatic example of a phenomenon we have already noted: for $\lambda < d$, non-negative symbols do not necessarily give rise to non-negative Toeplitz operators. Again, though, the conclusion that T_ϕ is negative for ϕ positive applies only when ϕ belongs to L^1 or L^2. For example, the constant function $\mathbf{1}$ always maps to the (positive!) identity operator, regardless of the value of λ.

5.3. Proofs

As motivation, we begin by computing the Hilbert–Schmidt norm of Toeplitz operators in the case $\lambda > d$. For any bounded measurable ϕ, we extend the Toeplitz operator T_ϕ to all of $L^2(\mathbb{B}^d, \mu_\lambda)$ by making it zero on the orthogonal complement of the holomorphic subspace. This extension is given by the formula $P_\lambda M_\phi P_\lambda$. Then the Hilbert–Schmidt norm of the operator T_ϕ on $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is the same as the Hilbert–Schmidt norm of the operator $P_\lambda M_\phi P_\lambda$ on $L^2(\mathbb{B}^d, \mu_\lambda)$. Since P_λ is computed as integration against the reproducing kernel, we may compute that

$$P_\lambda M_\phi P_\lambda f(z) = \int_{\mathbb{B}^d} \mathcal{K}_\phi(z, w) f(w)\, d\mu_\lambda(w),$$

where

$$\mathcal{K}_\phi(z, w) = \int_{\mathbb{B}^d} K(z, u)\phi(u)K(u, w)\, d\mu_\lambda(u).$$

If we can show that $\mathcal{K}_\phi$ is in $L^2(\mathbb{B}^d \times \mathbb{B}^d, \mu_\lambda \times \mu_\lambda)$, then it will follow by a standard result that T_ϕ is Hilbert–Schmidt, with Hilbert–Schmidt norm equal to the L^2 norm of $\mathcal{K}_\phi$. For sufficiently nice ϕ, we can compute the L^2 norm of $\mathcal{K}_\phi$ by rearranging the order of integration and using twice the reproducing identity $\int K(z, w)K(w, u)\ d\mu_\lambda(w) = K(z, u)$. (This identity reflects that $P_\lambda^2 = P_\lambda$.) This yields

$$\int_{\mathbb{B}^d \times \mathbb{B}^d} |\mathcal{K}_\phi(z, w)|^2\ d\mu_\lambda(z)\ d\mu_\lambda = \langle \phi, A\phi \rangle_{L^2(\mathbb{B}^d, \tau)},$$

where A_λ is the integral operator given by

$$\begin{aligned} A_\lambda\phi(z) &= c_\lambda^2 \int_{\mathbb{B}^d} |K(z,w)|^2 (1-|z|^2)^\lambda (1-|w|^2)^\lambda \phi(w)\ d\tau(w) \\ &= c_\lambda^2 \int_{\mathbb{B}^d} \left[\frac{(1-|z|^2)(1-|w|^2)}{(1-\bar{w}\cdot z)(1-\bar{z}\cdot w)}\right]^\lambda \phi(w)\, d\tau(w). \end{aligned} \tag{5.4}$$

In the case $d/2 < \lambda \leq d$, it no longer makes sense to express T_ϕ as $P_\lambda M_\phi P_\lambda$. Nevertheless, we can consider an operator A_λ defined by (5.4). Our goal is to show that for all $\lambda > d/2$, (1) A_λ is a bounded operator on $L^2(\mathbb{B}^d,\tau)$ and (2) if we define T_ϕ by (5.1), then the Hilbert–Schmidt norm of T_ϕ is given by $\langle \phi, A_\lambda \phi\rangle_{L^2(\mathbb{B}^d,\tau)}$. We will obtain similar results for all $\lambda > 0$ if $\phi \in L^1(\mathbb{B}^d,\tau)$.

Proof of Theorem 11. We give two proofs of this result; the first generalizes more easily to other bounded symmetric domains, whereas the second relates A_λ to the Laplacian for $\mathbb{B}^d$ (compare [13]).

First Proof. We let

$$F_\lambda(z,w) = c_\lambda^2 \left[\frac{(1-|z|^2)(1-|w|^2)}{(1-\bar{w}\cdot z)(1-\bar{z}\cdot w)}\right]^\lambda;$$

i.e., F_λ is the integral kernel of the operator A_λ. A key property of F_λ is its invariance under automorphisms: $F_\lambda(\psi(z),\psi(w)) = F_\lambda(z,w)$ for each automorphism (biholomorphism) ψ of $\mathbb{B}^d$ and all $z,w \in \mathbb{B}^d$. To establish the invariance of F_λ, let

$$f_\lambda(z) = c_\lambda^2 (1-|z|^2)^\lambda. \tag{5.5}$$

According to Lemma 1.2 of [Z2], $F_\lambda(z,w) = f_\lambda(\phi_w(z))$, where ϕ_w is an automorphism of $\mathbb{B}^d$ taking 0 to w and satisfying $\phi_w^2 = I$. Now, if ψ is any automorphism, the classification of automorphisms (Theorem 1.4 of [Z2]) implies that $\psi \circ \phi_w = \phi_{\psi(w)} \circ U$ for some unitary matrix U. From this we can obtain $\phi_{\psi(w)} = U \circ \phi_w \circ \psi^{-1}$, and so

$$f_\lambda(\phi_{\psi(w)}(\psi(z))) = f_\lambda(U(\phi_w(\psi^{-1}(\psi(z)))) = f_\lambda(\phi_w(z)),$$

i.e., $F_\lambda(\psi(z),\psi(w)) = F_\lambda(z,w)$.

The invariance of F_λ under automorphisms means that $A_\lambda\phi$ can be thought of as a convolution (over the automorphism group $PSU(d,1)$) of ϕ with the function f_λ. What this means is that

$$A_\lambda\phi(z) = \int_G f_\lambda(gh^{-1}\cdot 0)\phi(h\cdot 0)\ dh,$$

where $g \in G$ is chosen so that $g \cdot 0 = z$. Here $G = PSU(d,1)$ is the group of automorphisms of $\mathbb{B}^d$ (given by fractional linear transformations) and dh is an appropriately normalized Haar measure on G. Furthermore, $L^2(\mathbb{B}^d,\tau)$ can be identified with the right-K-invariant subspace of $L^2(G,dg)$, where $K := U(d)$ is the stabilizer of 0.

If $\lambda > d$, then f_λ is in $L^1(\mathbb{B}^d,\tau)$, in which case it is easy to prove that A_λ is bounded; see, for example, Theorem 2.4 in [5]. This argument does not work if

$\lambda \leq d$. Nevertheless, if $\lambda > d/2$, an easy computation shows that f_λ belongs to $L^2(\mathbb{B}^d, \tau)$ and also to $L^p(\mathbb{B}^d, \tau)$ for some $p < 2$. We could at this point appeal to a general result known as the Kunze–Stein phenomenon [24]. The result states that on connected semisimple Lie groups G with finite center (including $PSU(d, 1)$), convolution with a function in $L^p(G, dg)$, $p < 2$, is a bounded operator from $L^2(G, dg)$ to itself. (See [11] for a proof in this generality.) However, the proof of this result is simpler in the case we are considering, where the function in $L^p(G, dg)$ is bi-K-invariant and the other function is right-K-invariant. (In our case, the function in $L^p(G, dg)$ is the function $g \to f_\lambda(g \cdot 0)$ and the function in $L^2(G, dg)$ is $g \to \phi(g \cdot 0)$.) Using the Helgason Fourier transform along with its behavior under convolution with a bi-K-invariant function ([19, Lemma III.1.4]), we need only show that the spherical Fourier transform of f_λ is bounded. (Helgason proves Lemma III.1.4 under the assumption that the functions are continuous and of compact support, but the proof also applies more generally.) Meanwhile, standard estimates show that for every $\varepsilon > 0$, the spherical functions are in $L^{2+\varepsilon}(G/K)$, with $L^{2+\varepsilon}(G/K)$ norm bounded independent of the spherical function. (Specifically, in the notation of [18, Sect. IV.4], for all $\lambda \in \mathfrak{a}^*$, we have $|\phi_\lambda(g)| \leq \phi_0(g)$, and estimates on ϕ_0 (e.g., [1, Prop. 2.2.12]) show that ϕ_0 is in $L^{2+\varepsilon}$ for all $\varepsilon > 0$.)

Choosing ε so that $1/p + 1/(2 + \varepsilon) = 1$ establishes the desired boundedness.

Second proof. If $c_\lambda = 0$ (i.e., if $\lambda \in \mathbb{Z}$ and $\lambda \leq d$), then there is nothing to prove. Thus we assume c_λ is nonzero, in which case $c_{\lambda+1}$ is also nonzero. The invariance of F_λ under automorphisms together with the square-integrability of the function $(1 - |z|^2)^\lambda$ for $\lambda > d/2$ show that the integral defining $A_\lambda f(z)$ is absolutely convergent for all z.

We introduce the (hyperbolic) Laplacian Δ for $\mathbb{B}^d$, given by

$$\Delta = (1 - |z|^2) \sum_{j,k=1}^{d} (\delta_{jk} - \bar{z}_j z_k) \frac{\partial^2}{\partial \bar{z}_j \partial z_k}. \tag{5.6}$$

(This is a *negative* operator.) This operator commutes with the automorphisms of $\mathbb{B}^d$. It is known (e.g., [28]) that Δ is an unbounded self-adjoint operator on $L^2(\mathbb{B}^d, \tau)$, on the domain consisting of those f's in $L^2(\mathbb{B}^d, \tau)$ for which Δf in the distribution sense belongs to $L^2(\mathbb{B}^d, \tau)$. In particular, if $f \in L^2(\mathbb{B}^d, \tau)$ is C^2 and Δf in the ordinary sense belongs to $L^2(\mathbb{B}^d, \tau)$, then $f \in Dom(\Delta)$.

We now claim that

$$\Delta_z F_\lambda(z, w) = \lambda(\lambda - d)(F_\lambda(z, w) - F_{\lambda+1}(z, w)), \tag{5.7}$$

where Δ_z indicates that Δ is acting on the variable z with w fixed. Since Δ commutes with automorphisms, it again suffices to check this when $w = 0$, in which case it is a straightforward algebraic calculation. Suppose, then, that ϕ is a C^∞ function of compact support. In that case, we are free to differentiate under the integral to obtain

$$\Delta A_\lambda \phi = \lambda(\lambda - d) A_\lambda \phi - \lambda(\lambda - d) A_{\lambda+1} \phi. \tag{5.8}$$

Now, the invariance of F_λ tells us that $L^2(\mathbb{B}^d,\tau)$ norm of $F_\lambda(z,w)$ as a function of z is finite for all w and independent of w. Putting the L^2 norm inside the integral then shows that $A_\lambda\phi$ and $A_{\lambda+1}\phi$ are in $L^2(\mathbb{B}^d,\tau)$. This shows that $A_\lambda\phi$ is in $Dom(\Delta)$. Furthermore, the condition $\lambda > d/2$ implies that $\lambda(\lambda-d/2) > -d^2/4$. It is known that the L^2 spectrum of Δ is $(-\infty,-d^2/4]$. For general symmetric space of the noncompact type, the L^2 spectrum of the Laplacian is $(-\infty,-\left\|\rho\right\|^2]$, where ρ is half the sum of the positive (restricted) roots for G/K, counted with their multiplicity. In our case, there is one positive root α with multiplicity $(2d-2)$ and another positive root 2α with multiplicity 1. (See the entry for "A IV" in Table VI of Chapter X of [17].) Thus, $\rho = d\alpha$. It remains only to check that if the metric is normalized so that the Laplacian comes out as in (5.6), then $\left\|\alpha\right\|^2 = 1/4$. This is a straightforward but unilluminating computation, which we omit.

Since $\lambda(\lambda-d)$ is in the resolvent set of Δ, we may rewrite (5.8) as

$$A_\lambda\phi = -\lambda(\lambda-d)[\Delta-\lambda(\lambda-d)I]^{-1}A_{\lambda+1}\phi.$$

Suppose now that $\lambda+1 > d$, so that (as remarked above) $A_{\lambda+1}$ is bounded. Since $[\Delta-cI]^{-1}$ is a bounded operator for all c in the resolvent of Δ, we see that A_λ has a bounded extension from $C_c^\infty(\mathbb{B}^d)$ to $L^2(\mathbb{B}^d,\tau)$. Since the integral computing $A_\lambda\phi(z)$ is a continuous linear functional on $L^2(\mathbb{B}^d,\tau)$ (integration against an element of $L^2(\mathbb{B}^d,\tau)$), it is easily seen that this bounded extension coincides with the original definition of A_λ.

The above argument shows that A_λ is bounded if $\lambda > d/2$ and $\lambda+1 > d$. Iteration of the argument then shows boundedness for all $\lambda > d/2$. □

Proof of Theorem 12. We wish to show that for all $\lambda > d/2$, if ϕ is in $L^2(\mathbb{B}^d,\tau)$, then there is a unique Hilbert–Schmidt operator T_ϕ with matrix entries given in (5.1) for all polynomials, and furthermore, $\left\|T_\phi\right\|_{HS}^2 = \left\langle \phi, A_\lambda\phi\right\rangle_\lambda$. At the beginning of this section, we had an calculation of $\left\|T_\phi\right\|$ in terms of A_λ, but this argument relied on writing T_ϕ as $P_\lambda M_\phi P_\lambda$, which does not make sense for $\lambda \leq d$.

We work with an orthonormal basis for $H(\mathbb{B}^d,\lambda)$ consisting of normalized monomials, namely,

$$e_m(z) = z^m\sqrt{\frac{\Gamma(\lambda+|m|)}{m!\Gamma(\lambda)}},$$

for each multi-index m. Then we want to establish the existence of a Hilbert–Schmidt operator whose matrix entries in this basis are given by

$$a_{lm} := c_\lambda\int_{\mathbb{B}^d}\overline{e_l(z)}\phi(z)e_m(z)(1-|z|^2)^\lambda\ d\tau(z). \tag{5.9}$$

There will exist a unique such operator provided that $\sum_{l,m}|a_{lm}|^2 < \infty$.

If we assume, for the moment, that Fubini's Theorem applies, we obtain

$$\begin{aligned}
&\sum_{l,m} |a_{lm}|^2 \\
&= c_\lambda^2 \int_{\mathbb{B}^d} \int_{\mathbb{B}^d} \sum_{l,m} \frac{\Gamma(\lambda+|l|)}{l!\Gamma(\lambda)} \frac{\Gamma(\lambda+|m|)}{m!\Gamma(\lambda)} \bar{z}^l w^l z^m \bar{w}^m \\
&\times \phi(z)\overline{\phi(w)}(1-|z|^2)^\lambda (1-|w|^2)^\lambda \; d\tau(z) \; d\tau(w), \qquad (5.10)
\end{aligned}$$

where l and m range over all multi-indices of length d.

We now apply the binomial series

$$\frac{1}{(1-r)^\lambda} = \sum_{k=0}^\infty \binom{\lambda+k-1}{k} r^k$$

for $r \in \mathbb{C}$ with $|r| < 1$, where

$$\binom{\lambda+k-1}{k} = \frac{\Gamma(\lambda+k)}{k!\Gamma(\lambda)}.$$

(This is the so-called negative binomial series.) We apply this with $r = \sum_j \bar{z}_j w_j$, and we then apply the (finite) multinomial series to the computation of $(\bar{z} \cdot w)^k$. The result is that

$$\sum_l \frac{\Gamma(\lambda+|l|)}{l!\Gamma(\lambda)} \bar{z}^l w^l = \frac{1}{(1-\bar{z}\cdot w)^\lambda}, \qquad (5.11)$$

where the sum is over all multi-indices l. Applying this result, (5.10) becomes

$$\sum_{l,m} |a_{lm}|^2 = \langle \phi, A_\lambda \phi \rangle_\lambda, \qquad (5.12)$$

which is what we want to show.

Assume at first that ϕ is "nice," say, continuous and supported in a ball of radius $r < 1$. This ball has finite measure and ϕ is bounded on it. Thus, if we put absolute values inside the sum and integral on the right-hand side of (5.10), finiteness of the result follows from the absolute convergence of the series (5.11). Thus, Fubini's Theorem applies in this case.

Now for a general $\phi \in L^2(\mathbb{B}^d, \tau)$, choose ϕ_j converging to ϕ with ϕ_j "nice." Then (5.12) tells us that T_{ϕ_j} is a Cauchy sequence in the space of Hilbert–Schmidt operators, which therefore converges in the Hilbert–Schmidt norm to some operator T. The matrix entries of T_{ϕ_j} in the basis $\{e_m\}$ are by construction given by the integral in (5.9). The matrix entries of T are the limit of the matrix entries of T_{ϕ_j}, hence also given by (5.9), because e_l and e_m are bounded and $(1-|z|^2)^\lambda$ belongs to $L^2(\mathbb{B}^d, \tau)$ for $\lambda > d/2$.

We can now establish that (5.2) in Theorem 12 holds for all bounded holomorphic functions f and g in $H(\mathbb{B}^d, \lambda)$ by approximating these functions by polynomials. $\square$

Proof of Theorem 13. In the proof of Theorem 12, we did not use the assumption $\lambda > d/2$ until the step in which we approximated arbitrary functions in $L^2(\mathbb{B}^d, \tau)$ by "nice" functions. In particular, if ϕ is nice, then (5.9) makes sense for all $\lambda > 0$, and (5.12) still holds. Now, since $F_\lambda(z,w) = f_\lambda(\phi_w(z))$, where f_λ is given by (5.5), we see that $|F_\lambda(z,w)| \le c_\lambda^2$ for all $z, w \in \mathbb{B}^d$. Thus,

$$\langle \phi, A_\lambda \phi \rangle_\lambda \le c_\lambda^2 \, \|\phi\|^2_{L^1(\mathbb{B}^d,\tau)}$$

for all nice ϕ. An easy approximation argument then establishes the existence of a Hilbert–Schmidt operator with the desired matrix entries for all $\phi \in L^1(\mathbb{B}^d, \tau)$, with the desired estimate on the Hilbert–Schmidt norm. □

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